Weak type estimates of intrinsic square functions on the weighted Hardy spaces

Hua Wang *

School of Mathematical Sciences, Peking University, Beijing 100871, China

Abstract

In this paper, by using the atomic decomposition theory of weighted Hardy spaces, we will give some weighted weak type estimates for intrinsic square functions including the Lusin area function, Littlewood-Paley g-function and g^*_λ-function on these spaces.

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1. Introduction and preliminaries

First, let's recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [5]. Let w be a nonnegative, locally integrable function defined on $\mathbb{R}^n$, all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of the choice of Q.
For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C \operatorname*{ess\,inf}_{x\in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

*E-mail address: wanghua@pku.edu.cn.

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w.

Given a cube Q and $\lambda > 0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q. $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r. For a weight function w and a measurable set E, we set the weighted measure $w(E) = \int_E w(x)\,dx$.

We give the following result that will often be used in the sequel.

Lemma A ([2]). *Let $w \in A_p$, $p \geq 1$. Then, for any cube Q, there exists an absolute constant $C > 0$ such that*

$$w(2Q) \leq C w(Q).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda Q) \leq C \lambda^{np} w(Q),$$

where C does not depend on Q nor on λ.

Given a Muckenhoupt's weight function w on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

We also denote by $WL^p_w(\mathbb{R}^n)$ the weak weighted L^p space which is formed by all functions satisfying

$$\|f\|_{WL^p_w(\mathbb{R}^n)} = \sup_{\lambda > 0} \lambda \cdot w\big(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big)^{1/p} < \infty.$$

For any $0 < p < \infty$, the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ can be defined in terms of maximal functions. Let φ be a function in $\mathscr{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Set

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad t > 0,\ x \in \mathbb{R}^n.$$

We will define the maximal function $M_\varphi f(x)$ by

$$M_\varphi f(x) = \sup_{t > 0} |f * \varphi_t(x)|.$$

Then $H^p_w(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ for which $M_\varphi f \in L^p_w(\mathbb{R}^n)$ with $\|f\|_{H^p_w} = \|M_\varphi f\|_{L^p_w}$. For every $1 < p < \infty$, as in the unweighted case, we have $L^p_w(\mathbb{R}^n) = H^p_w(\mathbb{R}^n)$.

The real-variable theory of weighted Hardy spaces have been studied by many authors. In 1979, Garcia-Cuerva studied the atomic decomposition and the dual spaces of H^p_w for $0 < p \le 1$. In 2002, Lee and Lin gave the molecular characterization of H^p_w for $0 < p \le 1$, they also obtained the $H^p_w(\mathbb{R})$, $\frac{1}{2} < p \le 1$ boundedness of the Hilbert transform and the $H^p_w(\mathbb{R}^n)$, $\frac{n}{n+1} < p \le 1$ boundedness of the Riesz transforms. For the results mentioned above, we refer the readers to [1,3,6] for further details.

In this article, we will use Garcia-Cuerva's atomic decomposition theory for weighted Hardy spaces in [1,6]. We characterize weighted Hardy spaces in terms of atoms in the following way.

Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$ such that $w \in A_q$ with critical index q_w. Set $[\,\cdot\,]$ the greatest integer function. For $s \in \mathbb{Z}_+$ satisfying $s \ge [n(q_w/p-1)]$, a real-valued function $a(x)$ is called (p,q,s)-atom centered at x_0 with respect to w(or w-(p,q,s)-atom centered at x_0) if the following conditions are satisfied:

(a) $a \in L^q_w(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0,
(b) $\|a\|_{L^q_w} \le w(Q)^{1/q-1/p}$,
(c) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for every multi-index α with $|\alpha| \le s$.

Theorem B. *Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$ such that $w \in A_q$ with critical index q_w. For each $f \in H^p_w(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of w-$(p,q,[n(q_w/p-1)])$-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \le C\|f\|^p_{H^p_w}$ such that $f = \sum_j \lambda_j a_j$ both in the sense of distributions and in the H^p_w norm.*

2. The intrinsic square functions and our main results

The intrinsic square functions were first introduced by Wilson in [8] and [9], the so-called intrinsic square functions are defined as follows. For $0 < \alpha \le 1$, let $\mathcal{C}_\alpha$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x \in \mathbb{R}^n : |x| \le 1\}$, $\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$ and for all $x, x' \in \mathbb{R}^n$,

$$|\varphi(x) - \varphi(x')| \le |x - x'|^\alpha.$$

For $(y,t) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, we set

$$A_\alpha(f)(y,t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)|.$$

Then we define the intrinsic square function of f(of order α) by the formula

$$S_\alpha(f)(x) = \left(\iint_{\Gamma(x)} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}.$$

We can also define varying-aperture versions of $S_\alpha(f)$ by the formula

$$S_{\alpha,\beta}(f)(x) = \left(\iint_{\Gamma_\beta(x)} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma_\beta(x)$ is the usual cone of aperture $\beta > 0$:

$$\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}.$$

The intrinsic Littlewood-Paley g-function(could be viewed as "zero-aperture" version of $S_\alpha(f)$) and the intrinsic g^*_λ-function(could be viewed as "infinite aperture" version of $S_\alpha(f)$) will be defined respectively by

$$g_\alpha(f)(x) = \left(\int_0^\infty \Big(A_\alpha(f)(x,t)\Big)^2 \frac{dt}{t} \right)^{1/2}$$

and

$$g^*_{\lambda,\alpha}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

In [9], Wilson proved that the intrinsic square functions are bounded operators on the weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ for $1 < p < \infty$, namely, he showed the following result.

Theorem C. *Let $w \in A_p$, $1 < p < \infty$ and $0 < \alpha \le 1$. Then there exists a positive constant $C > 0$ such that*

$$\|S_\alpha(f)\|_{L^p_w} \le C\|f\|_{L^p_w}.$$

In [7], the authors considered some boundedness properties of intrinsic square functions on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $0 < p < 1$. Moreover, they gave the intrinsic square function characterizations of weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $0 < p < 1$. As a continuation of [7], the main purpose of this paper is to study their weak type estimates on these spaces.

In order to state our theorems, we need to introduce the Lipschitz space $Lip(\alpha,1,0)$ for $0 < \alpha \le 1$. Set $b_Q = \frac{1}{|Q|}\int_Q b(x)\,dx$.

$$Lip(\alpha,1,0) = \{b \in L_{loc}(\mathbb{R}^n) : \|b\|_{Lip(\alpha,1,0)} < \infty\},$$

where
$$\|b\|_{Lip(\alpha,1,0)} = \sup_Q \frac{1}{|Q|^{1+\alpha/n}} \int_Q |b(y) - b_Q|\, dy$$
and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Our main results are stated as follows.

Theorem 1. *Let $0 < \alpha < 1$, $p = n/(n+\alpha)$ and $w \in A_1$. Suppose that $f \in \big(Lip(\alpha,1,0)\big)^*$, then there exists a constant $C > 0$ independent of f such that*
$$\|g_\alpha(f)\|_{WL^p_w} \le C\|f\|_{H^p_w}.$$

Theorem 2. *Let $0 < \alpha < 1$, $p = n/(n+\alpha)$ and $w \in A_1$. Suppose that $f \in \big(Lip(\alpha,1,0)\big)^*$, then there exists a constant $C > 0$ independent of f such that*
$$\|S_\alpha(f)\|_{WL^p_w} \le C\|f\|_{H^p_w}.$$

Theorem 3. *Let $0 < \alpha < 1$, $p = n/(n+\alpha)$, $w \in A_1$ and $\lambda > 3 + (2\alpha)/n$. Suppose that $f \in \big(Lip(\alpha,1,0)\big)^*$, then there exists a constant C independent of f such that*
$$\|g^*_{\lambda,\alpha}(f)\|_{WL^p_w} \le C\|f\|_{H^p_w}.$$

Remark. Clearly, if for every $t > 0$, $\varphi_t \in \mathcal{C}_\alpha$, then we have $\varphi_t \in Lip(\alpha,1,0)$. Thus the intrinsic square functions are well defined for tempered distributions in $\big(Lip(\alpha,1,0)\big)^*$.

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

3. Proofs of Theorems 1 and 2

By adopting the same method given in [4, page 123], we can prove the following superposition principle on the weighted weak type estimates.

Lemma 3.1. *Let $w \in A_1$ and $0 < p < 1$. If a sequence of measurable functions $\{f_j\}$ satisfy*
$$\|f_j\|_{WL^p_w} \le 1 \quad \text{for all } j \in \mathbb{Z}$$
and
$$\sum_{j\in\mathbb{Z}} |\lambda_j|^p \le 1,$$
then we have
$$\Big\|\sum_{j\in\mathbb{Z}} \lambda_j f_j\Big\|^p_{WL^p_w} \le \frac{2-p}{1-p}.$$

Proof of Theorem 1. First we observe that for $w \in A_1$ and $p = n/(n+\alpha)$, then $[n(q_w/p-1)] = [\alpha] = 0$. By Theorem B and Lemma 3.1, it suffices to show that for any w-$(p,q,0)$-atom a, there exists a constant $C > 0$ independent of a such that $\|g_\alpha(a)\|_{WL^p_w} \le C$.

Let a be a w-$(p,q,0)$-atom with $supp\, a \subseteq Q = Q(x_0, r)$, and let $Q^* = 2\sqrt{n}Q$. For any given $\lambda > 0$, we write

$$\begin{aligned}
&\lambda^p \cdot w(\{x \in \mathbb{R}^n : |g_\alpha(a)(x)| > \lambda\}) \\
\le &\lambda^p \cdot w(\{x \in Q^* : |g_\alpha(a)(x)| > \lambda\}) + \lambda^p \cdot w(\{x \in (Q^*)^c : |g_\alpha(a)(x)| > \lambda\}) \\
= &I_1 + I_2.
\end{aligned}$$

Since $w \in A_1$, then $w \in A_q$ for $1 < q < \infty$. Applying Chebyshev's inequality, Hölder's inequality, Lemma A and Theorem C, we thus have

$$\begin{aligned}
I_1 &\le \int_{Q^*} |g_\alpha(a)(x)|^p w(x)\, dx \\
&\le \left(\int_{Q^*} |g_\alpha(a)(x)|^q w(x)\, dx\right)^{p/q} \left(\int_{Q^*} w(x)\, dx\right)^{1-p/q} \\
&\le \|g_\alpha(a)\|^p_{L^q_w} w(Q^*)^{1-p/q} \\
&\le C \cdot \|a\|^p_{L^q_w} w(Q)^{1-p/q} \\
&\le C.
\end{aligned} \tag{1}$$

We now turn to estimate I_2. For any $\varphi \in \mathcal{C}_\alpha$, $0 < \alpha < 1$, by the vanishing moment condition of atom a, we have

$$\begin{aligned}
\big|a * \varphi_t(x)\big| &= \left|\int_Q \big(\varphi_t(x-y) - \varphi_t(x-x_0)\big) a(y)\, dy\right| \\
&\le \int_Q \frac{|y-x_0|^\alpha}{t^{n+\alpha}} |a(y)|\, dy \\
&\le C \cdot \frac{r^\alpha}{t^{n+\alpha}} \int_Q |a(y)|\, dy.
\end{aligned} \tag{2}$$

For any fixed $q > 1$, we denote the conjugate exponent of q by $q' = q/(q-1)$. Hölder's inequality and the A_q condition yield

$$\begin{aligned}
\int_Q |a(y)|\, dy &\le \left(\int_Q |a(y)|^q w(y)\, dy\right)^{1/q} \left(\int_Q w(y)^{-1/(q-1)}\, dy\right)^{1/q'} \\
&\le C \cdot \|a\|_{L^q_w} \left(\frac{|Q|^q}{w(Q)}\right)^{1/q} \\
&\le C \cdot \frac{|Q|}{w(Q)^{1/p}}.
\end{aligned} \tag{3}$$

Observe that $supp\,\varphi \subseteq \{x \in \mathbb{R}^n : |x| \le 1\}$, then for any $y \in Q$, $x \in (Q^*)^c$, we have $t \ge |x-y| \ge |x-x_0| - |y-x_0| \ge \frac{|x-x_0|}{2}$. Substituting the above inequality (3) into (2), we thus obtain

$$\begin{aligned}
\big|g_\alpha(a)(x)\big|^2 &= \int_0^\infty \Big(\sup_{\varphi\in\mathcal{C}_\alpha} \big|a*\varphi_t(x)\big|\Big)^2 \frac{dt}{t}\\
&\le C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^2 \int_{\frac{|x-x_0|}{2}}^\infty \frac{dt}{t^{2(n+\alpha)+1}}\\
&\le C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^2 \frac{1}{|x-x_0|^{2n+2\alpha}}\\
&\le C\left(\frac{1}{w(Q)^{1/p}}\right)^2.
\end{aligned}$$

Set $Q_0^* = Q$, $Q_1^* = Q^*$ and $Q_k^* = (Q_{k-1}^*)^*, k = 2,3,\dots$. Following the same lines as above, we can also show that for any $x \in (Q_k^*)^c$, then

$$\big|g_\alpha(a)(x)\big| \le C \cdot \frac{1}{w(Q_{k-1}^*)^{1/p}} \quad k = 1,2,\dots.$$

If $\{x \in (Q^*)^c : |g_\alpha(a)(x)| > \lambda\} = \emptyset$, then the inequality

$$I_2 \le C$$

holds trivially.
If $\{x \in (Q^*)^c : |g_\alpha(a)(x)| > \lambda\} \ne \emptyset$. For $p = n/(n+\alpha)$, it is easy to verify that

$$\lim_{k\to\infty} \frac{1}{w(Q_k^*)^{1/p}} = 0.$$

Then for any fixed $\lambda > 0$, we are able to find a maximal positive integer K such that

$$\lambda < C \cdot \frac{1}{w(Q_K^*)^{1/p}}.$$

Therefore

$$\begin{aligned}
I_2 &\le \lambda^p \cdot \sum_{k=1}^K w\big(\{x \in Q_{k+1}^* \backslash Q_k^* : |g_\alpha(a)(x)| > \lambda\}\big)\\
&\le C \cdot \frac{1}{w(Q_K^*)} \sum_{k=1}^K w(Q_{k+1}^*)\\
&\le C.
\end{aligned} \tag{4}$$

Combining the above inequality (4) with (1) and taking the supremum over all $\lambda > 0$, we complete the proof of Theorem 1. $\square$

Proof of Theorem 2. The proof is almost the same. We only point out the main differences. For any given $\lambda > 0$, we write

$$\begin{aligned}&\lambda^p \cdot w(\{x \in \mathbb{R}^n : |S_\alpha(a)(x)| > \lambda\})\\ \leq&\lambda^p \cdot w(\{x \in Q^* : |S_\alpha(a)(x)| > \lambda\}) + \lambda^p \cdot w(\{x \in (Q^*)^c : |S_\alpha(a)(x)| > \lambda\})\\ =&J_1 + J_2.\end{aligned}$$

Using the same arguments as in the proof of Theorem 1, we can prove

$$J_1 \leq C.$$

To estimate J_2, we note that $z \in Q$, $x \in (Q^*)^c$, then $|z - x_0| \leq \frac{|x-x_0|}{2}$. Furthermore, when $|x - y| < t$ and $|y - z| < t$, then we deduce

$$2t > |x - z| \geq |x - x_0| - |z - x_0| \geq \frac{|x - x_0|}{2}.$$

By using the inequalities (2) and (3), we thus obtain

$$\begin{aligned}|S_\alpha(a)(x)|^2 &= \iint_{\Gamma(x)} \left(\sup_{\varphi \in \mathcal{C}_\alpha} |a * \varphi_t(y)|\right)^2 \frac{dydt}{t^{n+1}}\\ &\leq C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^2 \int_{\frac{|x-x_0|}{4}}^{\infty} \int_{|y-x|<t} \frac{dydt}{t^{2n+2\alpha+n+1}}\\ &\leq C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^2 \frac{1}{|x - x_0|^{2n+2\alpha}}\\ &\leq C\left(\frac{1}{w(Q)^{1/p}}\right)^2,\end{aligned}$$

which is equivalent to

$$|S_\alpha(a)(x)| \leq C \cdot \frac{1}{w(Q)^{1/p}}.$$

The rest of the proof is exactly the same as that of Theorem 1, we can get

$$J_2 \leq C.$$

This completes the proof of Theorem 2. $\square$

4. Proof of Theorem 3

Before proving our main theorem, we need to establish the following lemma.

Lemma 4.1. *Let $w \in A_1$ and $0 < \alpha \le 1$. Then for every $\lambda > 1$, we have*

$$\|g^*_{\lambda,\alpha}(a)\|_{L^2_w} \le C\|a\|_{L^2_w}.$$

Proof. From the definition, we readily see that

$$\begin{aligned}
\big(g^*_{\lambda,\alpha}(a)(x)\big)^2 &= \iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}} \\
&= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}} \\
&\quad + \sum_{k=1}^\infty \int_0^\infty \int_{2^{k-1}t\le|x-y|<2^k t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} (A_\alpha(a)(y,t))^2 \frac{dydt}{t^{n+1}} \\
&\le C\left[S_\alpha(a)(x)^2 + \sum_{k=1}^\infty 2^{-k\lambda n} S_{\alpha,2^k}(a)(x)^2\right]. \qquad (5)
\end{aligned}$$

We are now going to estimate $\int_{\mathbb{R}^n} |S_{\alpha,2^k}(a)(x)|^2 w(x)\,dx$ for $k = 1, 2, \dots$. Fubini theorem and Lemma A imply

$$\begin{aligned}
&\int_{\mathbb{R}^n} |S_{\alpha,2^k}(a)(x)|^2 w(x)\,dx \\
=& \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{n+1}_+} \Big(A_\alpha(a)(y,t)\Big)^2 \chi_{|x-y|<2^k t} \frac{dydt}{t^{n+1}}\right) w(x)\,dx \\
=& \int_{\mathbb{R}^{n+1}_+} \left(\int_{|x-y|<2^k t} w(x)\,dx\right) \Big(A_\alpha(a)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \\
\le& C \cdot 2^{kn} \int_{\mathbb{R}^{n+1}_+} \left(\int_{|x-y|<t} w(x)\,dx\right) \Big(A_\alpha(a)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \qquad (6) \\
=& C \cdot 2^{kn} \|S_\alpha(a)\|^2_{L^2_w}.
\end{aligned}$$

Since $w \in A_1$, then $w \in A_2$. Therefore, by using Theorem C and the above inequality (6), we thus obtain

$$\begin{aligned}
&\|g^*_{\lambda,\alpha}(a)\|^2_{L^2_w} \\
\le& C\left(\int_{\mathbb{R}^n} |S_\alpha(a)(x)|^2 w(x)\,dx + \sum_{k=1}^\infty 2^{-k\lambda n} \int_{\mathbb{R}^n} |S_{\alpha,2^k}(a)(x)|^2 w(x)\,dx\right) \\
\le& C\left(\|S_\alpha(a)\|^2_{L^2_w} + \sum_{k=1}^\infty 2^{-k\lambda n} \cdot 2^{kn} \|S_\alpha(a)\|^2_{L^2_w}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le C \cdot \|a\|_{L^2_w}^2 \Big(1 + \sum_{k=1}^{\infty} 2^{-k\lambda n} \cdot 2^{kn}\Big)\\
&\le C \cdot \|a\|_{L^2_w}^2.
\end{aligned}
$$

We are done. $\square$

We are now in a position to give the proof of Theorem 3.

Proof of Theorem 3. As in the proof of Theorem 1, we write

$$
\begin{aligned}
&\sigma^p \cdot w(\{x \in \mathbb{R}^n : |g^*_{\lambda,\alpha}(a)(x)| > \sigma\})\\
\le&\sigma^p \cdot w(\{x \in Q^* : |g^*_{\lambda,\alpha}(a)(x)| > \sigma\}) + \sigma^p \cdot w(\{x \in (Q^*)^c : |g^*_{\lambda,\alpha}(a)(x)| > \sigma\})\\
=&K_1 + K_2.
\end{aligned}
$$

Note that $\lambda > 3 + (2\alpha)/n > 1$. Applying Chebyshev's inequality, Hölder's inequality, Lemma A and Lemma 4.1, we thus have

$$
\begin{aligned}
K_1 &\le \int_{Q^*} |g^*_{\lambda,\alpha}(a)(x)|^p w(x)\,dx\\
&\le \left(\int_{Q^*} |g^*_{\lambda,\alpha}(a)(x)|^2 w(x)\,dx\right)^{p/2} \left(\int_{Q^*} w(x)\,dx\right)^{1-p/2}\\
&\le \|g^*_{\lambda,\alpha}(a)\|_{L^2_w}^p w(Q^*)^{1-p/2}\\
&\le C \cdot \|a\|_{L^2_w}^p w(Q)^{1-p/2}\\
&\le C.
\end{aligned}
$$

We now turn to deal with K_2. In the proof of Theorem 2, we have already showed

$$
|S_\alpha(a)(x)|^2 \le C \left(\frac{1}{w(Q)^{1/p}}\right)^2. \tag{7}
$$

For any given $(y,t) \in \Gamma_{2^k}(x)$, $x \in (Q^*)^c$, then a simple calculation shows that $t \ge \frac{|x-x_0|}{2^{k+2}}$, $k \in \mathbb{Z}_+$. Hence, by the estimates (2) and (3), we can get

$$
\begin{aligned}
|S_{\alpha,2^k}(a)(x)|^2 &= \iint_{\Gamma_{2^k}(x)} \Big(\sup_{\varphi \in \mathcal{C}_\alpha} |a * \varphi_t(y)|\Big)^2 \frac{dydt}{t^{n+1}}\\
&\le C \left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^2 \int_{\frac{|x-x_0|}{2^{k+2}}}^{\infty} \int_{|y-x|<2^k t} \frac{dydt}{t^{2n+2\alpha+n+1}}\\
&\le C \cdot 2^{k(3n+2\alpha)} \left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^2 \frac{1}{|x-x_0|^{2n+2\alpha}}\\
&\le C \cdot 2^{k(3n+2\alpha)} \left(\frac{1}{w(Q)^{1/p}}\right)^2.
\end{aligned}
\tag{8}
$$

It follows immediately from the inequalities (5), (7) and (8) that

$$\begin{aligned}\left(g^*_{\lambda,\alpha}(a)(x)\right)^2 &\leq C\left(\frac{1}{w(Q)^{1/p}}\right)^2\left(1+\sum_{k=1}^{\infty}2^{-k\lambda n}\cdot 2^{k(3n+2\alpha)}\right)\\ &\leq C\left(\frac{1}{w(Q)^{1/p}}\right)^2,\end{aligned}$$

where the last series is convergent since $\lambda > 3 + (2\alpha)/n$. Again, the rest of the proof is exactly the same as that of Theorem 1, we finally obtain

$$K_2 \leq C.$$

Therefore, we conclude the proof of Theorem 3. $\square$

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